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Nonpermanence of the change in four-momentum of the gravitational field of an isolated radiating source

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Abstract. The well known use of the Einstein energy pseudo-tensor density ℓ^i_k for determining the rate of change of four-momentum, by gravitational radiation, of an isolated cohesive source in motion contains an important assumption: the total potential four-momentum of the gravitational field, represented by the integral of ℓ^4_k over all space, undergoes no secular variation. A mathematical justification of this assumption is presented here. However, an example is discussed in which consideration must be given to the variation of the total potential three-momentum of the field, despite the nonpermanence of this variation: namely the cyclic motion of the newtonian centre of mass of a bounded, cohesive, purely rotating system.

Finally it is shown that replacement of the Einstein pseudo-tensor density by that of Landau and Lifshitz has no effect on the main results of this paper.

1. Introduction

The Einstein pseudo-tensor density ℓ^i_k of potential (or gravitational) energy, which is connected with the material energy tensor density $\mathscr{T}^i_k$ by the conservation law (Eddington 1924 § 59, Tolman 1934 § 87, Weber 1961 § 6.1)†

$$(\mathscr{T}^i_k + \ell^i_k)_{,i} = 0 \tag{1.1}$$

has been used in several works on general relativity to show that, according to the linear approximation to the Einstein field equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -8\pi T_{ik} \tag{1.2}$$

gravitational waves carry away four-momentum from an emitter (Bonnor and Rotenberg 1961, Weber 1961 §§ 7.5 and 7.6, Landau and Lifshitz 1962 § 104, Papapetrou 1962). This is explained in the following way. Let S be a two-dimensional sphere with centre the origin O (of x_α) and of radius r, containing space volume V and including the source. Then, integrating equation (1.1) over V and using Gauss' theorem we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (\mathscr{T}^4_k + \ell^4_k)\,\mathrm{d}v = -\int_S (\mathscr{T}^\alpha_k + \ell^\alpha_k) n_\alpha \,\mathrm{d}S \tag{1.3}$$

† In this paper, a Latin index runs from 1 to 4, a Greek one from 1 to 3; the comma notation, indicating partial differentiation, and the summation convention are assumed for both types of indices. The coordinates used throughout this paper are (pseudo-)galilean coordinates $x_i = (x_\alpha, t) = (x, y, z, t)$.

where $n_\alpha \stackrel{\text{def}}{=} x_\alpha/r$ are the components of the outward space normal to S. Since $\mathscr{T}^i_k = 0$ on S, we obtain

$$\frac{\mathrm{d}J_k}{\mathrm{d}t} \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_V (\mathscr{T}^4_k + t^4_k)\,\mathrm{d}v = -\int_S t^\alpha_k n_\alpha\,\mathrm{d}S. \tag{1.4}$$

This is the well known integral theorem referring to the four-momentum of matter and its gravitational field: the left-hand side represents the rate of change of four-momentum J_k of the source plus field inside S; this is given by the right-hand side, which represents the rate of flow of four-momentum of the field into S.

To apply equation (1.4) to an emitter of gravitational radiation, the right-hand side is calculated for the appropriate wave field. With contributions of order r^{-1} ignored for a large sphere S, the result is (Landau and Lifshitz 1962 § 104)

$$\frac{\mathrm{d}J_4}{\mathrm{d}t} = -\frac{1}{45}Q'''_{\beta\gamma}Q'''_{\beta\gamma} \tag{1.5}$$

and (Papapetrou 1962)

$$\frac{\mathrm{d}J_\alpha}{\mathrm{d}t} = \frac{4}{315}\{Q'''_{\alpha\beta}(4M''_{\gamma\gamma:\beta} - 6M''_{\beta\gamma:\gamma}) + Q'''_{\beta\gamma}(11M''_{\beta\gamma:\alpha} - 6M''_{\alpha\beta:\gamma})\}. \tag{1.6}$$

Here,

$$Q_{\alpha\beta} \stackrel{\text{def}}{=} \int_V (3x_\alpha x_\beta - \delta_{\alpha\beta}x_\gamma x_\gamma)T_{44}\,\mathrm{d}v \tag{1.7}$$

are the quadrupole moments about the coordinate planes $x_\alpha = 0$, which, along with

$$M_{\alpha\beta:\gamma} \stackrel{\text{def}}{=} \int_V x_\gamma T_{\alpha\beta}\,\mathrm{d}v \tag{1.8}$$

are to be calculated for retarded time $u = t - r$; a prime denotes differentiation with respect to u. The formulae (1.5) and (1.6) are based on the quadrupole and octupole wave solutions of the linear approximation to equation (1.2) (obtained in § 2).

It has been the practice to ignore the contribution

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V t^4_k\,\mathrm{d}v \tag{1.9}$$

for large r, on the left of equation (1.4) and, consequently, to regard the formulae (1.5) and (1.6) as referring to the change in four-momentum of the source. This was done on the assumption that the total potential four-momentum of the gravitational field, represented by

$$\int_{\text{all space}} t^4_k\,\mathrm{d}v \tag{1.10}$$

undergoes no secular change (Eddington 1924 § 59, Weber 1961 § 7.5)†. The main object of this paper is to establish mathematically the truth of this assumption for an isolated cohesive source in any motion.

To achieve this we first derive the external solution, for the source, of the linear approximation to equation (1.2); this is done in § 2. Then in § 3, by introducing what is

† Contrast this assumption with the criticism made by Peres (1960) on a proof by Infeld (1959) that gravitational radiation does not exist.

known as Synge's argument and applying it to the solution, we set up an indirect method of calculating the integral (1.10) to show that it suffers no secular variation. As explained there, direct computation of this integral would be an almost impossible task. Included in § 4 is a brief discussion of the cyclic motion of the newtonian centre of mass of an isolated, coherent, purely rotating source. The purpose of this is to illustrate that not every time is it permissible to disregard the variation of the total potential three-momentum of the field, despite the cyclic property of this variation. To avoid such errors, a suitable correction term (namely the expression (4.3)) is added to the formula on the right of equation (1.6). The paper ends (§ 5) with a proof that the main results are unaltered when the Einstein pseudo-tensor density is replaced by that of Landau and Lifshitz.

2. Solution of the linearized field equations

We outline here a derivation of a solution of the linear approximation to the field equations (1.2) appropriate to an isolated coherent distribution of matter, its centre of mass chosen as the origin O of space coordinates x_α.

For weak fields we may write

$$g_{ik} = \eta_{ik}+\gamma_{ik}, \qquad \eta_{ik} = \eta^{ik} \stackrel{\text{def}}{=} \operatorname{diag}(-1,\,-1,\,-1,\,+1) \tag{2.1}$$

where γ_{ik} are small. Introduce ϕ_{ik} by

$$\phi_{ik} = \gamma_{ik}-\tfrac{1}{2}\eta_{ik}\eta^{ab}\gamma_{ab} \leftrightarrow \gamma_{ik} = \phi_{ik}-\tfrac{1}{2}\eta_{ik}\eta^{ab}\phi_{ab} \tag{2.2}$$

and select (pseudo-)galilean coordinates x_i satisfying the harmonic condition

$$\eta^{ab}\phi_{ia,b} = 0. \tag{2.3}$$

The linear approximation to equation (1.2) then reduces to the wave equation (Eddington 1924 § 57, Landau and Lifshitz 1962 § 101)

$$\eta^{ab}\phi_{ik,ab} = -16\pi T_{ik}. \tag{2.4}$$

Of equations (2.4) and (2.3), the solution in Kirchhoff form for outgoing waves is (Eddington 1924 § 57)

$$\phi_{ik} = -4\int_V r^{*-1}T_{ik}(\tilde{x}_\alpha, t-r^*)\,\mathrm{d}v, \qquad \eta^{ab}T_{ia,b} = 0 \tag{2.5}$$

in which V is a fixed space volume containing all the sources of the field and r^* is the distance of the point $\tilde{\mathrm{P}}(\tilde{x}_\alpha)$ ($\tilde{x}_\alpha = \tilde{x}, \tilde{y}, \tilde{z}$), associated with the space element $\mathrm{d}v = \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{z}$ of integration, from the field point $\mathrm{P}(x_\alpha)$ of interest. The second of equations (2.5), which expresses the conservation law of four-momentum in the linear approximation, ensures that the first of equations (2.5) satisfies equation (2.3) as well as equation (2.4) (Eddington 1924 § 57).

It is useful to have the multipole expansion of this solution, got by expanding the integrand on the right of the first of equations (2.5) by the Taylor theorem so that r, the fixed distance OP, occurs in place of r^*. The expansion involves moments of T_{ik} of all orders about the coordinate planes, namely

$$I_{ik:\sigma\rho\tau\ldots}(t) \stackrel{\text{def}}{=} \int_V x_\sigma x_\rho x_\tau \ldots T_{ik}(x_\alpha, t)\,\mathrm{d}v, \qquad \eta^{ab}T_{ia,b} = 0. \tag{2.6}$$

Indeed, let $m \stackrel{\text{def}}{=} I_{44}$ be the total mass of the sources and a be a constant with the dimension of length, and let these parameters m, a be defined in the newtonian sense but in relativistic units employed in this work. Introduce the following specific moments $h_{ik:\sigma\rho\tau\ldots}$, unaffected by any change of units in m or a:

$$\begin{aligned}
h_{\alpha\beta:\sigma_1\sigma_2\ldots\sigma_s} &\stackrel{\text{def}}{=} m^{-1}a^{-s-2}I_{\alpha\beta:\sigma_1\sigma_2\ldots\sigma_s} \\
h_{\alpha4:\sigma_1\sigma_2\ldots\sigma_s} &\stackrel{\text{def}}{=} m^{-1}a^{-s-1}I_{\alpha4:\sigma_1\sigma_2\ldots\sigma_s} \\
h_{44:\sigma_1\sigma_2\ldots\sigma_s} &\stackrel{\text{def}}{=} m^{-1}a^{-s}I_{44:\sigma_1\sigma_2\ldots\sigma_s}.
\end{aligned} \tag{2.7}$$

Then the multipole expansion turns out to be (see Rotenberg 1964 appendix A, 1972)

$$\phi_{ik} = m\overset{1}{\phi}_{ik} \tag{2.8}$$

with $\overset{1}{\phi}_{ik}$ given, explicitly up to order a^3, by

$$\overset{1}{\phi}_{\alpha\beta} = -4a^2r^{-1}h_{\alpha\beta} - 4a^3n_\sigma(r^{-1}h'_{\alpha\beta:\sigma} + r^{-2}h_{\alpha\beta:\sigma}) + \mathrm{O}(a^4) \tag{2.9}$$

$$\begin{aligned}
\overset{1}{\phi}_{\alpha4} = {} & -4a^2n_\sigma(r^{-1}h'_{\alpha4:\sigma} + r^{-2}h_{\alpha4:\sigma}) \\
& -2a^3\{n_\sigma n_\rho r^{-1}h''_{\alpha4:\sigma\rho} + (3n_\sigma n_\rho - \delta_{\sigma\rho})(r^{-2}h'_{\alpha4:\sigma\rho} + r^{-3}h_{\alpha4:\sigma\rho})\} + \mathrm{O}(a^4)
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\overset{1}{\phi}_{44} = {} & -4r^{-1} - 2a^2\{n_\sigma n_\rho r^{-1}h''_{44:\sigma\rho} + (3n_\sigma n_\rho - \delta_{\sigma\rho})(r^{-2}h'_{44:\sigma\rho} + r^{-3}h_{44:\sigma\rho})\} \\
& -\tfrac{2}{3}a^3\{n_\sigma n_\rho n_\tau r^{-1}h'''_{44:\sigma\rho\tau} + 3n_\sigma(2n_\rho n_\tau - \delta_{\rho\tau})r^{-2}h''_{44:\sigma\rho\tau} \\
& +3n_\sigma(5n_\rho n_\tau - 3\delta_{\rho\tau})(r^{-3}h'_{44:\sigma\rho\tau} + r^{-4}h_{44:\sigma\rho\tau})\} + \mathrm{O}(a^4).
\end{aligned} \tag{2.11}$$

Here $n_\alpha \stackrel{\text{def}}{=} x_\alpha/r$; $h_{ik:\sigma\rho\tau\ldots}$ are to be evaluated for time $u = t - r$; and a prime denotes differentiation with respect to u.

This solution, which is an external solution (for the sources) of the linear approximation to equation (1.2) or to

$$R_{ik} = 0 \tag{2.12}$$

is known as the multipole wave solution of the linear approximation; its 2^s pole contribution ($s = 0, 1, 2, \ldots$) is formed by the coefficients of a^s on the right-hand sides of equations (2.9) to (2.11). The dipole wave is absent from the solution (a well known result); the lowest wave-like terms are those of the quadrupole wave, involving a^2.

From equation (2.8) it is clear that the linear approximation to equation (2.12), being linear in ϕ_{ik} (and their derivatives), is linear in m. Thus the multipole wave solution (2.8) to (2.11) satisfies that contribution in equation (2.12) which is linear in m.

To reduce considerably the calculations required in § 4, we apply a standard treatment (outlined below) on the linearized conservation equations in equations (2.6), which may be split up as

$$T_{\alpha\beta,\beta} = T_{\alpha4,4}, \qquad T_{4\beta,\beta} = T_{44,4}. \tag{2.13}$$

This treatment on equations (2.13) generates an infinite sequence of relations among $h_{ik:\sigma\rho\tau\ldots}$, of which the leading ones are (Landau and Lifshitz 1962 § 104, Papapetrou 1962, Rotenberg 1964 appendix A)

$$h_{44} = 1 \tag{2.14}$$

$$h_{\alpha4} = h_{44:\alpha} = 0 \tag{2.15}$$

$$h_{\alpha\beta} = -h'_{4\alpha:\beta} = \tfrac{1}{2}h''_{44:\alpha\beta}, \qquad h_{4(\alpha:\beta)} = -h'_{44:\alpha\beta} \tag{2.16}$$

$$h_{\alpha(\beta:\gamma)} = -h'_{4\alpha:\beta\gamma}, \qquad h_{(\alpha\beta:\gamma)} = \tfrac{1}{2}h''_{44:\alpha\beta\gamma}, \qquad h_{4(\alpha:\beta\gamma)} = -h'_{44:\alpha\beta\gamma} \tag{2.17}$$

the notations

$$A_{(\alpha\beta)} \stackrel{\text{def}}{=} A_{\alpha\beta}+A_{\beta\alpha}, \qquad A_{(\alpha\beta\gamma)} \stackrel{\text{def}}{=} A_{\alpha\beta\gamma}+A_{\beta\gamma\alpha}+A_{\gamma\alpha\beta} \tag{2.18}$$

involving cyclic permutations of subscripts, have been employed in equations (2.16) and (2.17). In equations (2.9) to (2.11) we use the first two relations of equations (2.16) and of equations (2.17) and the notations

$$\begin{aligned} &m_{\alpha\beta} \stackrel{\text{def}}{=} h_{44:\alpha\beta}, \qquad m_{\alpha:\beta} \stackrel{\text{def}}{=} h_{4\alpha:\beta} \\ &m_{\alpha\beta:\gamma} \stackrel{\text{def}}{=} h_{\alpha\beta:\gamma}, \qquad m_{\alpha:\beta\gamma} \stackrel{\text{def}}{=} h_{4\alpha:\beta\gamma}, \qquad m_{\alpha\beta\gamma} \stackrel{\text{def}}{=} h_{44:\alpha\beta\gamma}; \end{aligned} \tag{2.19}$$

the multipole wave solution becomes

$$\overset{1}{\phi}_{\alpha\beta} = -2a^2r^{-1}m''_{\alpha\beta}-4a^3n_\sigma(r^{-1}m'_{\alpha\beta:\sigma}+r^{-2}m_{\alpha\beta:\sigma})+O(a^4) \tag{2.20}$$

$$\begin{aligned} \overset{1}{\phi}_{\alpha4} = {} & 2a^2n_\sigma(r^{-1}m''_{\alpha\sigma}-2r^{-2}m_{\alpha:\sigma}) \\ & +2a^3\{2n_\sigma n_\rho r^{-1}m'_{\alpha\sigma:\rho}+(3n_\sigma n_\rho-\delta_{\sigma\rho})(2r^{-2}m_{\alpha\sigma:\rho}-r^{-3}m_{\alpha:\sigma\rho})\}+O(a^4) \end{aligned} \tag{2.21}$$

$$\begin{aligned} \overset{1}{\phi}_{44} = {} & -4r^{-1}-2a^2\{n_\sigma n_\rho r^{-1}m''_{\sigma\rho}+(3n_\sigma n_\rho-\delta_{\sigma\rho})(r^{-2}m'_{\sigma\rho}+r^{-3}m_{\sigma\rho})\} \\ & -2a^3\{2n_\sigma n_\rho n_\tau r^{-1}m'_{\sigma\rho:\tau}+2n_\sigma(2n_\rho n_\tau-\delta_{\rho\tau})r^{-2}m_{(\sigma\rho:\tau)} \\ & +n_\sigma(5n_\rho n_\tau-3\delta_{\rho\tau})(r^{-3}m'_{\sigma\rho\tau}+r^{-4}m_{\sigma\rho\tau})\}+O(a^4). \end{aligned} \tag{2.22}$$

In these, primes accompanying the functions $m_{\alpha\beta},\dots$ introduced from equations (2.19) mean differentiation with respect to $u = t-r$, at which time these functions are to be calculated.

This solution will be needed in §§ 3 and 4 (and appendix 1). There it is assumed that an appropriate complete external solution of equation (2.12) is expansible as an infinite series in ascending powers of m,

$$\phi_{ik} = \sum_{p=1}^{\infty} m^p \overset{p}{\phi}_{ik} \qquad (\overset{p}{\phi}_{ik} \text{ independent of } m) \tag{2.23}$$

with the leading contribution $\overset{1}{\phi}_{ik}$ to ϕ_{ik} given by the multipole wave solution (2.20) to (2.22) of the linear approximation (to equation (2.12)).

To conclude this section we derive the relations (2.17) as an illustration of the standard treatment on the conservation equations (2.13) for yielding an infinite sequence of relationships involving $h_{ik:\sigma\rho\tau\dots}$. Multiplying the first of equations (2.13) by $x_\gamma x_\delta$ and integrating over any fixed space volume V enclosing the source distribution, we have

$$\begin{aligned} \frac{d}{dt}\int_V x_\gamma x_\delta T_{\alpha4}\,dv &= \int_V x_\gamma x_\delta T_{\alpha4,4}\,dv = \int_V x_\gamma x_\delta T_{\alpha\beta,\beta}\,dv \\ &= \int_V (x_\gamma x_\delta T_{\alpha\beta})_{,\beta}\,dv - \int_V (x_\delta T_{\alpha\gamma}+x_\gamma T_{\alpha\delta})\,dv. \end{aligned} \tag{2.24}$$

By virtue of Gauss' theorem the first integral on the extreme right of equation (2.24) vanishes; so, using the first of equations (2.6), equations (2.7) and the first of equations (2.18) in equation (2.24), we obtain the first of the relations (2.17). Similarly, multiplying the second of equations (2.13) by $x_\gamma x_\delta x_\epsilon$, integrating over V and utilizing Gauss' theorem, the first of equations (2.6), equations (2.7) and the second of equations (2.18), we get the third of the relations (2.17). The first and third of the relations (2.17), when combined, yield the second of these relations. Similar methods are employed to establish the relations (2.14) to (2.16) and, indeed, the set of relations among the moments $h_{ik:\sigma\rho\tau\ldots}$ of any specified order.

3. Variation of the total potential four-momentum

To calculate the total potential four-momentum (1.10) of the gravitational field we shall use an argument due originally to Synge (1960, § IV-6) and clearly explained by Bonnor (1959). This argument applies to approximate solutions of the field equations for any gravitating source, and for a bounded cohesive source (referred to as Σ in this section) it is as follows.

Consider any approximate solution of equation (2.12) relevant to the source Σ. Substituting the solution into

$$G^i_k \stackrel{\text{def}}{=} R^i_k - \tfrac{1}{2}\delta^i_k R = -8\pi T^i_k \tag{3.1}$$

we can get a value for the energy tensor T^i_k corresponding to it. This will not vanish anywhere, in contrast with the case of an appropriate exact solution in which T^i_k is zero everywhere outside Σ. Thus for the approximate solution, a continuous distribution of matter Σ' will be present throughout space–time, and this distribution combined with Σ may be regarded as representing the 'source' of the approximate solution. To put it more concisely, the approximate solution of equation (2.12) may be taken as an exact solution of equation (3.1) corresponding to the extended source $\Sigma+\Sigma'$.

Let us now use this argument for the external solution (2.8). For the source Σ it satisfies the linear approximation, in which the four-momentum of Σ is conserved. Since the solution satisfies the contribution in equation (2.12) that is linear in m, it follows that if we insert the solution in equation (3.1), we will have for the energy tensor density an expression in the form of an infinite series in ascending powers of m starting with the term in m^2. This expansion may be written as

$$\mathscr{T}^i_k = -\frac{1}{8\pi}\sum_{p=2}^{\infty} m^p \overset{p}{G}{}^i_k \qquad (\overset{p}{G}{}^i_k \text{ independent of } m) \tag{3.2}$$

where the leading contribution $\overset{2}{G}{}^i_k$, calculated in appendix 1, is the coefficient of m^2 in the Einstein tensor G^i_k. Outside Σ the metric (2.8), by Synge's argument, represents exactly the continuous distribution Σ' described by the energy tensor density (3.2); it has four-momentum flowing out of a large sphere S, centre O, at the rate

$$\frac{\mathrm{d}J_k}{\mathrm{d}t} = \int_S \mathscr{T}^\alpha_k n_\alpha \,\mathrm{d}S = -\frac{1}{8\pi}\sum_{p=2}^{\infty} m^p \int_S \overset{p}{G}{}^\alpha_k n_\alpha \,\mathrm{d}S. \tag{3.3}$$

Consequently, to maintain the metric field (2.8), and thus to maintain the constancy of four-momentum of Σ, matter Σ' must be extracted out of S from the neighbourhood of Σ with four-momentum flowing across S at the rate (3.3). However, in the actual physical situation no such matter Σ' exists, whence it follows that the four-momentum of the central source Σ must increase (or, to be more exact, change) at the rate $(\mathrm{d}J_k/\mathrm{d}t)_\Sigma$ given by equation (3.3). This result may be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathscr{T}^4_k\,\mathrm{d}v = \left(\frac{\mathrm{d}J_k}{\mathrm{d}t}\right)_\Sigma = -\frac{m^2}{8\pi}\int_S \overset{2}{G}{}^\alpha_k n_\alpha\,\mathrm{d}S + \mathrm{O}(m^3) \tag{3.4}$$

V being the volume contained by S. Hence from equation (1.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V t^4_k\,\mathrm{d}v = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathscr{T}^4_k\,\mathrm{d}v - \int_S t^\alpha_k n_\alpha\,\mathrm{d}S = -\frac{m^2}{16\pi}\int_S (-2\overset{2}{G}{}^\alpha_k + 16\pi \overset{2}{t}{}^\alpha_k) n_\alpha\,\mathrm{d}S + \mathrm{O}(m^3) \tag{3.5}$$

in which $\overset{2}{t}{}^i_k$ is the coefficient of m^2 in t^i_k. So, a long calculation gives (see appendix 1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{all space}} t^4_k\,\mathrm{d}v = -\frac{m^2}{64\pi}\left\{\lim_{r\to\infty}\int_S (2\psi^{ab}\psi_{ab} - \psi\psi)\lambda_k\,\mathrm{d}S\right\}_{,44} + \mathrm{O}(m^3): \tag{3.6}$$

in this

$$\psi_{ik} \overset{\text{def}}{=} \overset{1}{\phi}_{ik} + 4r^{-1}\delta_{i4}\delta_{k4} \tag{3.7}$$

$$\lambda_i \overset{\text{def}}{=} u_{,i} = (-n_\alpha, 1), \qquad \lambda^i \overset{\text{def}}{=} \eta^{ia}\lambda_a = (n_\alpha, 1) \tag{3.8}$$

so that

$$\lambda_i = \eta_{ia}\lambda^a, \qquad \lambda^a\lambda_a = 0 \tag{3.9}$$

and

$$\psi \overset{\text{def}}{=} -\eta^{ab}\psi_{ab}, \qquad \psi^{ik} \overset{\text{def}}{=} \eta^{ia}\eta^{kb}\psi_{ab}. \tag{3.10}$$

As explained in appendix 1, equation (3.6) holds not only for the approximate metric (2.8) for Σ but also for the exact metric (2.23) for Σ.

From the formula (3.6) we can immediately establish the nonpermanent nature of the time variation of the integral (1.10). The expression involving m^2 on the right of equation (3.6) is an exact (second) time derivative. Thus, up to order m^2 at least, equation (3.6) can be integrated directly with respect to t to give†

$$\int_{\text{all space}} t^4_k\,\mathrm{d}v = \text{constant} - \frac{m^2}{64\pi}\lim_{r\to\infty}\int_S (2\psi^{ab}\psi_{ab} - \psi\psi)_{,4}\lambda_k\,\mathrm{d}S + \mathrm{O}(m^3). \tag{3.11}$$

It can be shown that the r^{-1} terms of the nonvanishing 2^s pole wave contributions in ψ_{ik}, those of the quadrupole wave onwards, contain as factors time derivatives of the specific moments $h_{ik:\sigma\rho\tau\ldots}$ (see Rotenberg 1972); for the quadrupole and octupole waves at least, this is evident from equations (3.7) and (2.20) to (2.22). Hence, the r^{-1} contribution in ψ_{ik} and its time derivatives undergo no secular time variation and, consequently, the same applies to the expression involving m^2 on the right of equation (3.11). Thus we have established that the integral (1.10) suffers no permanent change. Although this

† To obtain directly the formula (3.11) for the expression (1.10), without the aid of Synge's argument, would require knowledge of a suitable solution of the linear approximation for the entire space–time, including the region of the source Σ. Finding such a solution would necessitate the almost impossible task of matching an internal solution with an external solution for a moving system, not spherically symmetric. In contrast, the above use of Synge's argument for deriving the formula (3.11) involves only the external solution ψ_{ik} of the linear approximation.

result is perhaps only true to order m^2, it is certainly a more accurate statement than the formulae (1.5) and (1.6) for the rate of change in four-momentum of the source plus field; the latter have been derived only as far as the quadrupole–quadrupole and quadrupole–octupole terms, respectively, of the m^2 contribution to the left of equation (1.4).

In contrast with the foregoing result, the total four-momentum J_k of the source plus field may change permanently. This is due to the fact that its rate of change, given (on account of equations (1.4) and (A.15)) by

$$\frac{\mathrm{d}J_k}{\mathrm{d}t} = -\frac{m^2}{64\pi}\lim_{r\to\infty}\int_S (2\psi^{ab}{}_{,4}\psi_{ab,4} - \psi_{,4}\psi_{,4})\lambda_k\,\mathrm{d}S + \mathrm{O}(m^3) \tag{3.12}$$

is not in the form of an exact time derivative, even in its m^2 contribution. So, J_k contains a time-integral expression, which in many cases undergoes secular variation despite the fact that $\psi_{ik,4}$ certainly does not. To confirm this we note from equation (1.5) that, for $k = 4$, the quadrupole–quadrupole part of this integral expression is

$$-\tfrac{1}{45}\int Q'''_{\alpha\beta}Q'''_{\alpha\beta}\,\mathrm{d}t \tag{3.13}$$

which numerically increases steadily with time, except for those types of motion of the source with the rare property that the third time derivatives of the quadrupole moments $Q_{\alpha\beta}$ are identically zero.

4. Cyclic motion of the newtonian centre of mass of a purely rotating system

In spite of the main result of the previous section, there are important examples in which the total potential three-momentum of the gravitational field, represented by the integral (1.10) ($k = \alpha$), must be taken into account in the evaluation of the material three-momentum of the source. One such example is the secular motion of the newtonian centre of mass of a bounded, coherent, purely rotating system—a circular recoil caused by the three-momentum flow of emitted gravitational radiation. Obviously from equation (1.4), a mere cyclic time variation in the integral (1.10) is sufficient to have an effect on the magnitude of this circular recoil.

To illustrate this, let us consider for simplicity a rigid rod spinning in the xy plane with angular velocity ω about its supposedly fixed centre of mass, chosen as the origin O. In a previous paper (Rotenberg 1968) it has been shown that the centre of mass of the rod moves with the same angular velocity ω along a fixed circle, centre O and radius

$$\delta = \frac{172}{35}\frac{\overset{2}{I}\overset{3}{I}\omega^5}{m} \tag{4.1}$$

where m is the mass of the rod and $\overset{n}{I}$ is its nth moment about O. The part $(13/129)\delta$ of this radius arises from due consideration of the integral (1.10). Only the remaining part comes from the use of the formula (1.6).

To avoid such errors resulting from formula (1.6) in related examples, an appropriate correction term should be added to the right of equation (1.6). By virtue of equations (3.6) to (3.10), (2.20) to (2.22), (2.19), (2.7), (2.6), (1.7) and (1.8), this correction term, given by

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{all space}} t^4_\alpha\,\mathrm{d}v \tag{4.2}$$

turns out to be

$$-\tfrac{4}{315}\{Q''_{\alpha\beta}(4M'_{\gamma\gamma:\beta}-6M'_{\beta\gamma:\gamma})+Q''_{\beta\gamma}(11M'_{\beta\gamma:\alpha}-6M'_{\alpha\beta:\gamma})\}''. \tag{4.3}$$

5. Use of the Landau and Lifshitz pseudo-tensor†

We conclude this paper by showing that the use of the Landau and Lifshitz pseudo-tensor density ${}_{\mathrm{L}}t^{ik}$ instead of the Einstein pseudo-tensor density t^i_k (which in this section will be written as ${}_{\mathrm{E}}t_k{}^i$ to avoid confusion) does not affect the results of the previous two sections. It is sufficient to prove that

$${}_{\mathrm{L}}t^{ik}-{}_{\mathrm{E}}t^{ik} = m^2\mathrm{O}(r^{-3})+\mathrm{O}(m^3) \tag{5.1}$$

it being understood that the rule for raising and lowering indices of tensors applies to pseudo-tensors as well.

In terms of the tensor density

$$\mathscr{T}^{iklm} \overset{\mathrm{def}}{=} \mathfrak{g}^{il}\mathfrak{g}^{km}-\mathfrak{g}^{im}\mathfrak{g}^{kl} \tag{5.2}$$

of weight $+2$ we have

$$16\pi(\mathscr{T}^{ik}+{}_{\mathrm{L}}t^{ik}) = (-g)^{-1/2}\mathscr{T}^{akbi}{}_{,ba} \tag{5.3}$$

$$16\pi(\mathscr{T}_k{}^i+{}_{\mathrm{E}}t_k{}^i) = \{g_{kl}(-g)^{-1/2}\mathscr{T}^{aibl}{}_{,b}\}_{,a} \tag{5.4}$$

(see Møller 1958, Landau and Lifshitz 1962 § 100, Cornish 1964). Applying the same rule for raising and lowering indices for pseudo-tensors as for tensors we obtain from equation (5.4)

$$\begin{aligned}16\pi(\mathscr{T}^{ik}+{}_{\mathrm{E}}t^{ik}) &= g^{im}\{g_{ml}(-g)^{-1/2}\mathscr{T}^{akbl}{}_{,b}\}_{,a}\\ &= g^{im}[\{g_{ml}(-g)^{-1/2}\}_{,a}\mathscr{T}^{akbl}{}_{,b}+g_{ml}(-g)^{-1/2}\mathscr{T}^{akbl}{}_{,ba}]\\ &= g^{im}\{g_{ml}(-g)^{-1/2}\}_{,a}\mathscr{T}^{akbl}{}_{,b}+\delta^i_l(-g)^{-1/2}\mathscr{T}^{akbl}{}_{,ba}\\ &= g^{im}\{g_{ml}(-g)^{-1/2}\}_{,a}\mathscr{T}^{akbl}{}_{,b}+(-g)^{-1/2}\mathscr{T}^{akbi}{}_{,ba}\end{aligned}$$

so that by virtue of equation (5.3)

$$16\pi(\mathscr{T}^{ik}+{}_{\mathrm{E}}t^{ik}) = g^{im}\{g_{ml}(-g)^{-1/2}\}_{,a}\mathscr{T}^{akbl}{}_{,b}+16\pi(\mathscr{T}^{ik}+{}_{\mathrm{L}}t^{ik}).$$

This gives

$$16\pi({}_{\mathrm{L}}t^{ik}-{}_{\mathrm{E}}t^{ik}) = -g^{im}\{g_{ml}(-g)^{-1/2}\}_{,a}\mathscr{T}^{akbl}{}_{,b} \tag{5.5}$$

which is an homogeneous quadratic expression in the derivatives of the metric, as expected. In appendix 2 it is shown that this expression vanishes to order m^2 and r^{-2}, and so equation (5.1) is established. Hence substitution of the Einstein pseudo-tensor density with that of Landau and Lifshitz would yield the same results as those of the previous two sections.

Appendix 1. The Einstein tensor and the Einstein pseudo-tensor

We outline the calculations leading from equation (3.5) to equation (3.6).

† This section has been introduced as the result of a suggestion by a referee.

First we derive, for the linear approximation to the metric in the form (2.8), suitable expressions for the Einstein tensor G^i_k and the Einstein pseudo-tensor density $\mathfrak{t}^i_k$ applicable to any isolated cohesive mechanical system. We substitute equation (2.1) (with $\overset{1}{\gamma}_{ik}$ given in terms of $\overset{1}{\phi}_{ik}$ by the second of equations (2.2) and equation (2.8)) into the formulae

$$R_{ik} = \Gamma^a_{ia,k} - \Gamma^a_{ik,a} + \Gamma^a_{ib}\Gamma^b_{ka} - \Gamma^a_{ik}\Gamma^b_{ab} \tag{A.1}$$

$$16\pi\mathfrak{t}^i_k = \mathfrak{g}^{ab}{}_{,k}\Gamma^i_{ab} - \mathfrak{g}^{ai}{}_{,k}\Gamma^b_{ab} + \delta^i_k\mathfrak{g}^{lm}(\Gamma^a_{lb}\Gamma^b_{ma} - \Gamma^a_{lm}\Gamma^b_{ab}) \tag{A.2}$$

for the Ricci tensor and the pseudo-tensor density, the formula for $\mathfrak{t}^i_k$ being equivalent to that given by Eddington (1924 § 59) and Tolman (1934 § 87). Then, on using the harmonic and wave equations (2.3) and (2.4) ($T_{ik} = 0$) whenever possible, we obtain

$$\begin{aligned}R_{ik} = m^2[&\tfrac{1}{4}\overset{1}{\phi}\overset{1}{\phi}_{,ik} + \tfrac{1}{8}\overset{1}{\phi}_{,i}\overset{1}{\phi}_{,k}\\ &+\eta^{ab}\{\tfrac{1}{4}(\overset{1}{\phi}_{ia}\overset{1}{\phi}_{,kb} + \overset{1}{\phi}_{ka}\overset{1}{\phi}_{,ib}) + \overset{1}{\phi}_{,a}(\tfrac{1}{4}\overset{1}{\phi}_{ib,k} + \tfrac{1}{4}\overset{1}{\phi}_{kb,i} - \tfrac{1}{2}\overset{1}{\phi}_{ik,b} - \tfrac{1}{8}\eta_{ik}\overset{1}{\phi}_{,b})\}\\ &+\eta^{ab}\eta^{cd}\{\overset{1}{\phi}_{ac}(\tfrac{1}{2}\overset{1}{\phi}_{ib,kd} + \tfrac{1}{2}\overset{1}{\phi}_{kb,id} - \tfrac{1}{2}\overset{1}{\phi}_{ik,bd} - \tfrac{1}{2}\overset{1}{\phi}_{bd,ik} - \tfrac{1}{4}\eta_{ik}\overset{1}{\phi}_{,bd})\\ &-\tfrac{1}{4}\overset{1}{\phi}_{ac,i}\overset{1}{\phi}_{bd,k} + \tfrac{1}{2}\overset{1}{\phi}_{ia,c}(\overset{1}{\phi}_{kd,b} - \overset{1}{\phi}_{kb,d})\}] + \mathrm{O}(m^3)\end{aligned} \tag{A.3}$$

$$16\pi\mathfrak{t}^i_k = m^2(\breve{u}^i_k - \tfrac{1}{2}\delta^i_k\breve{u}^a_a) + \mathrm{O}(m^3) \tag{A.4}$$

where

$$\breve{u}^i_k \overset{\text{def}}{=} \eta^{il}\{-\tfrac{1}{4}\overset{1}{\phi}_{,k}\overset{1}{\phi}_{,l} + \eta^{ab}\eta^{cd}\overset{1}{\phi}_{ac,k}(\tfrac{1}{2}\overset{1}{\phi}_{bd,l} - \overset{1}{\phi}_{bl,d})\} \tag{A.5}$$

and

$$\overset{1}{\phi} \overset{\text{def}}{=} -\eta^{ab}\overset{1}{\phi}_{ab}. \tag{A.6}$$

If contributions of order r^{-3} are neglected for large r, the above expressions (A.3) and (A.5) can be simplified in the following way.

It is easily verified from the form of equations (2.20) to (2.22) that

$$\overset{1}{\phi}_{ik,a} = \lambda_a\overset{1}{\phi}_{ik,4} + \mathrm{O}(r^{-2}), \qquad \overset{1}{\phi}_{ik,ab} = \lambda_a\lambda_b\overset{1}{\phi}_{ik,44} + \mathrm{O}(r^{-2}), \qquad \ldots, \tag{A.7}$$

in which λ_i, along with λ^i, are given by equations (3.8). The coordinate condition (2.3) thus yields

$$\lambda^b\overset{1}{\phi}_{ab,4} = \mathrm{O}(r^{-2}). \tag{A.8}$$

Integrating this with respect to t we find

$$\lambda^b\overset{1}{\phi}_{ab} = -4r^{-1}\delta_{a4} + \mathrm{O}(r^{-2}) \tag{A.9}$$

noting that all terms in r^{-1} on the right of the multipole wave solution (2.20) to (2.22) depend on time except the leading term on the right of equation (2.22), since these terms do not occur in the corresponding static solution of the linear approximation for a static source. Using equations (A.6) to (A.9) and equations (3.9) in equations (A.3) and (A.5) and then the result in

$$G^i_k = R^i_k - \tfrac{1}{2}\delta^i_k R \tag{A.10}$$

and equation (A.4), we obtain for the Einstein tensor and the pseudo-tensor density

$$\begin{aligned} G^i_k = {} & m^2[\lambda^i\lambda_k(\tfrac{1}{4}\overset{1}{\phi}\overset{1}{\phi}_{,44}+\tfrac{1}{8}\overset{1}{\phi}_{,4}\overset{1}{\phi}_{,4}-\tfrac{1}{2}\overset{1}{\phi}{}^{ab}\overset{1}{\phi}_{ab,44}-\tfrac{1}{4}\overset{1}{\phi}{}^{ab}{}_{,4}\overset{1}{\phi}_{ab,4}) \\ & +r^{-1}\eta^{ia}\{2(\overset{1}{\phi}_{ak}-\lambda_a\overset{1}{\phi}_{k4}-\lambda_k\overset{1}{\phi}_{a4})+(\eta_{ak}-\lambda_a\delta_{k4}-\lambda_k\delta_{a4})\overset{1}{\phi}\}_{,44}+\mathrm{O}(r^{-3})] \\ & +\mathrm{O}(m^3) \end{aligned} \tag{A.11}$$

$$16\pi\mathfrak{t}^i_k = m^2\{\lambda^i\lambda_k(\tfrac{1}{2}\overset{1}{\phi}{}^{ab}{}_{,4}\overset{1}{\phi}_{ab,4}-\tfrac{1}{4}\overset{1}{\phi}_{,4}\overset{1}{\phi}_{,4})+\mathrm{O}(r^{-3})\}+\mathrm{O}(m^3) \tag{A.12}$$

where

$$\overset{1}{\phi}{}^{ik} \overset{\mathrm{def}}{=} \eta^{ia}\eta^{kb}\overset{1}{\phi}_{ab}. \tag{A.13}$$

Using equation (3.7), the second of equations (3.8), equations (3.10), (A.6) and (A.13) in equations (A.11) and (A.12), we get for the coefficients $\overset{2}{G}{}^i_k$ and $\overset{2}{\mathfrak{t}}{}^i_k$ of m^2 in G^i_k and $\mathfrak{t}^i_k$, respectively,

$$-2\overset{2}{G}{}^i_k = \lambda^i\lambda_k(\psi^{ab}\psi_{ab,44}+\tfrac{1}{2}\psi^{ab}{}_{,4}\psi_{ab,4}-\tfrac{1}{2}\psi\psi_{,44}-\tfrac{1}{4}\psi_{,4}\psi_{,4})-2L^i_k+\mathrm{O}(r^{-3}) \tag{A.14}$$

$$16\pi\overset{2}{\mathfrak{t}}{}^i_k = \lambda^i\lambda_k(\tfrac{1}{2}\psi^{ab}{}_{,4}\psi_{ab,4}-\tfrac{1}{4}\psi_{,4}\psi_{,4})+\mathrm{O}(r^{-3}) \tag{A.15}$$

where

$$L^i_k \overset{\mathrm{def}}{=} r^{-1}\eta^{ia}\{2(\psi_{ak}-\lambda_a\psi_{k4}-\lambda_k\psi_{a4}+\lambda_a\lambda_k\psi_{44})+(\eta_{ak}-\lambda_a\delta_{k4}-\lambda_k\delta_{a4}+\lambda_a\lambda_k)\psi\}_{,44}. \tag{A.16}$$

With the help of equations (A.8) and (3.7) to (3.9) it can be shown that

$$L^\alpha_\beta n_\alpha = \mathrm{O}(r^{-3}), \qquad L^i_4 = 0. \tag{A.17}$$

Hence, insertion of equations (A.14) and (A.15) in equation (3.5) and use of the second of equations (3.8) and equation (A.13) gives equation (3.6). Formula (3.6) is true for both the approximate metric (2.8) and the exact metric (2.23), because the difference between the values of $\mathfrak{t}^i_k$ corresponding to these two metrics is of order m^3, the right of equation (A.2) consisting of products of $\phi_{ik,l} = \mathrm{O}(m)$.

Appendix 2. Relationship between the Einstein pseudo-tensor and the Landau and Lifshitz pseudo-tensor

Our aim here is to establish equation (5.1) by showing that the expression on the right of equation (5.5) is zero to order m^2 and r^{-2}.

The first of equations (2.1) may be written as

$$g_{ik} = \eta_{ik}+m\overset{1}{\gamma}_{ik}+\mathrm{O}(m^2) \tag{A.18}$$

where $\overset{1}{\gamma}_{ik}$ is the coefficient of m in γ_{ik}; then the first of equations (2.2) yields

$$\overset{1}{\phi}_{ik} = \overset{1}{\gamma}_{ik}-\tfrac{1}{2}\eta_{ik}\eta^{ab}\overset{1}{\gamma}_{ab}. \tag{A.19}$$

From equation (A.19) it readily follows that

$$\eta^{ia}\eta^{kb}\overset{1}{\phi}_{ab} = \eta^{ia}\eta^{kb}\overset{1}{\gamma}_{ab}-\tfrac{1}{2}\eta^{ik}\eta^{ab}\overset{1}{\gamma}_{ab}. \tag{A.20}$$

It can easily be shown from equation (A.18) that

$$g^{ik} = \eta^{ik} - m\eta^{ia}\eta^{kb}\overset{1}{\gamma}_{ab} + \mathrm{O}(m^2) \tag{A.21}$$

$$g = -1 - m\eta^{ab}\overset{1}{\gamma}_{ab} + \mathrm{O}(m^2) \tag{A.22}$$

the latter equation giving

$$(-g)^{\pm 1/2} = 1 \pm \tfrac{1}{2}m\eta^{ab}\overset{1}{\gamma}_{ab} + \mathrm{O}(m^2) \qquad \text{(signs corresponding).} \tag{A.23}$$

Hence

$$\begin{aligned}\mathfrak{g}^{ik} = g^{ik}(-g)^{1/2} &= \eta^{ik} - m(\eta^{ia}\eta^{kb}\overset{1}{\gamma}_{ab} - \tfrac{1}{2}\eta^{ik}\eta^{ab}\overset{1}{\gamma}_{ab}) + \mathrm{O}(m^2)\\ &= \eta^{ik} - m\eta^{ia}\eta^{kb}\overset{1}{\phi}_{ab} + \mathrm{O}(m^2)\end{aligned} \tag{A.24}$$

by virtue of equation (A.20). From equations (A.18) and (A.23),

$$\begin{aligned}g_{ik}(-g)^{-1/2} &= \eta_{ik} + m(\overset{1}{\gamma}_{ik} - \tfrac{1}{2}\eta_{ik}\eta^{ab}\overset{1}{\gamma}_{ab}) + \mathrm{O}(m^2)\\ &= \eta_{ik} + m\overset{1}{\phi}_{ik} + \mathrm{O}(m^2)\end{aligned} \tag{A.25}$$

on account of equation (A.19). Because of equation (2.3), equation (A.24) yields

$$\mathfrak{g}^{ik}{}_{,k} = \mathrm{O}(m^2). \tag{A.26}$$

So it follows from equation (5.2) that

$$\mathscr{T}^{akbl}{}_{,b} = (\mathfrak{g}^{ab}\mathfrak{g}^{kl} - \mathfrak{g}^{al}\mathfrak{g}^{kb})_{,b} = \mathfrak{g}^{ab}\mathfrak{g}^{kl}{}_{,b} - \mathfrak{g}^{kb}\mathfrak{g}^{al}{}_{,b} + \mathrm{O}(m^2). \tag{A.27}$$

Consequently, it is readily found from equation (A.24) that

$$\mathscr{T}^{akbl}{}_{,b} = m\eta^{ld}(\eta^{kb}\eta^{ac} - \eta^{kc}\eta^{ab})\overset{1}{\phi}_{cd,b} + \mathrm{O}(m^2). \tag{A.28}$$

Equation (A.25) gives

$$\{g_{ml}(-g)^{-1/2}\}_{,a} = m\overset{1}{\phi}_{ml,a} + \mathrm{O}(m^2). \tag{A.29}$$

Multiplying the two equations (A.28) and (A.29) together and making use of the first of equations (A.7) and the fact that $\overset{1}{\phi}_{ik}$ is of order r^{-1}, we obtain

$$\begin{aligned}\{g_{ml}(-g)^{-1/2}\}_{,a}\mathscr{T}^{akbl}{}_{,b} &= m^2\{\eta^{ld}(\eta^{kb}\eta^{ac} - \eta^{kc}\eta^{ab})\lambda_a\lambda_b\overset{1}{\phi}_{cd,4}\overset{1}{\phi}_{ml,4} + \mathrm{O}(r^{-3})\} + \mathrm{O}(m^3)\\ &= m^2\eta^{ld}\overset{1}{\phi}_{ml,4}\{\lambda^k(\lambda^c\overset{1}{\phi}_{cd,4}) - \eta^{kc}(\lambda^b\lambda_b)\overset{1}{\phi}_{cd,4}\} + m^2\mathrm{O}(r^{-3}) + \mathrm{O}(m^3)\end{aligned}$$

by virtue of the second of the definitions (3.8). On account of equation (A.8) and the second of equations (3.9) this result leads to

$$\{g_{ml}(-g)^{-1/2}\}_{,a}\mathscr{T}^{akbl}{}_{,b} = m^2\mathrm{O}(r^{-3}) + \mathrm{O}(m^3) \tag{A.30}$$

so that the right-hand side of equation (5.5) vanishes to order m^2 and r^{-2}; this implies the validity of equation (5.1).

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